

# ON THE SECOND BOUNDARY VALUE PROBLEM FOR MONGE-AMPÈRE TYPE EQUATIONS AND OPTIMAL TRANSPORTATION

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**ABSTRACT.** This paper is concerned with the existence of globally smooth solutions for the second boundary value problem for Monge-Ampère type equations and the application to regularity of potentials in optimal transportation. The cost functions satisfy a weak form of the condition A3, which was introduced in a recent paper with Xi-nan Ma in conjunction with interior regularity. Consequently they include the quadratic cost function case of Caffarelli and Urbas as well as the various examples in the earlier work. The approach is through the derivation of global estimates for second derivatives of solutions.

## 1. Introduction

This paper is concerned with the global regularity of solutions of the second boundary value problem for equations of Monge-Ampère type and its application to the regularity of potentials in optimal transportation problems with non-quadratic cost functions.

The Monge-Ampère equations under consideration have the general form

$$(1.1) \quad \det\{D^2u - A(\cdot, u, Du)\} = B(\cdot, u, Du),$$

where  $A$  and  $B$  are given  $n \times n$  matrix and scalar valued function defined on  $\Omega \times \mathbb{R} \times \mathbb{R}^n$ , where  $\Omega$  is a domain in Euclidean  $n$ -space,  $\mathbb{R}^n$ . We use  $(x, z, p)$  to denote points in  $\Omega \times \mathbb{R} \times \mathbb{R}^n$  so that  $A(x, z, p) \in \mathbb{R}^n \times \mathbb{R}^n$ ,  $B(x, z, p) \in \mathbb{R}$  and  $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ . The equation (1.1) will be elliptic, (degenerate elliptic), with respect to a solution  $u \in C^2(\Omega)$  whenever

$$(1.2) \quad D^2u - A(\cdot, u, Du) > 0 \quad (\geq 0),$$

whence also  $B > 0$  ( $\geq 0$ ).

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This research was supported by the Australian Research Council.

A particular form of (1.1) arises from the prescription of the Jacobian determinant of a mapping  $Tu$  defined by

$$(1.3) \quad Tu = Y(\cdot, u, Du),$$

where  $Y$  is a given vector valued function on  $\Omega \times \mathbb{R} \times \mathbb{R}^n$ , namely

$$(1.4) \quad \det DY(\cdot, u, Du) = \psi(\cdot, u, Du).$$

Assuming that the matrix

$$(1.5) \quad Y_p = [D_{p_j} Y^i]$$

is non-singular, we may write (1.4) in the form (1.1), that is,

$$(1.6) \quad \det\{D^2u + Y_p^{-1}(Y_x + Y_z \otimes Du)\} = \frac{\psi}{|\det Y_p|},$$

for degenerate elliptic solutions  $u$ .

The *second boundary value problem* for equation (1.4) is to prescribe the image

$$(1.7) \quad Tu(\Omega) = \Omega^*,$$

where  $\Omega^*$  is a given domain in  $\mathbb{R}^n$ . When  $Y$  and  $\psi$  are independent of  $z$  and  $\psi$  is separable in the sense that

$$(1.8) \quad \psi(x, p) = f(x)/g \circ Y(x, p)$$

for positive  $f, g \in L^1(\Omega), L^1(\Omega^*)$  respectively, then a necessary condition for the existence of an elliptic solution, for which the mapping  $T$  is a diffeomorphism, to the second boundary value problem (1.4) (1.7) is the *mass balance* condition

$$(1.9) \quad \int_{\Omega} f = \int_{\Omega^*} g.$$

The second boundary value problem (1.4) (1.7) arises naturally in optimal transportation. Here we are given a cost function  $c : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  and the vector field  $Y$  is generated by the equation

$$(1.10) \quad c_x(x, Y(x, p)) = p,$$

which we assume to be uniquely solvable for  $p \in \mathbb{R}^n$ , with non-vanishing determinant, that is

$$(1.11) \quad \det c_{x,y}(x, y) \neq 0$$

for all  $x, y \in \Omega \times \Omega^*$ . Using the notation

$$(1.12) \quad c_{ij\dots,kl\dots} = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \dots \frac{\partial}{\partial y_k} \frac{\partial}{\partial y_l} \dots c$$

we have

$$(1.13) \quad Y_p(x, p) = [c^{i,j}(x, Y(x, p))],$$

where  $[c^{i,j}]$  is the inverse of  $[c_{i,j}]$ . The corresponding Monge-Ampère equation can now be written as

$$(1.14) \quad \det\{D^2u - c_{xx}(\cdot, Y(\cdot, Du))\} = |\det c_{x,y}|\psi,$$

that is in the form (1.1) with

$$(1.15) \quad \begin{aligned} A(x, z, p) &= c_{xx}(x, Y(x, p)), \\ B(x, z, p) &= |\det c_{x,y}(x, Y(x, p))|\psi(x, z, p). \end{aligned}$$

In the case of the (quadratic) cost function

$$(1.16) \quad c(x, y) = x \cdot y,$$

we have

$$(1.17) \quad Y(x, p) = p, \quad Tu = Du,$$

and equation (1.14) reduces to the standard Monge-Ampère equation

$$(1.18) \quad \det D^2u = \psi.$$

For this case global regularity of solutions was proved by Delanoë [D], Caffarelli [C2] and Urbas [U1], with (conditional) interior regularity shown earlier by Caffarelli [C1]. In this paper we will prove global estimates and regularity under corresponding conditions. In particular, we will assume that the cost function  $c \in C^4(\mathbb{R}^n \times \mathbb{R}^n)$  satisfies the following conditions:

(A1) For each  $p, q \in \mathbb{R}^n$ , there exists unique  $y = Y(x, p)$ ,  $x = X(q, y)$  such that

$$\begin{aligned} c_x(x, y) &= p \quad \forall x \in \Omega, \\ c_y(x, y) &= q \quad \forall y \in \Omega^*. \end{aligned}$$

$$(A2) \quad \det c_{x,y}(x, y) \neq 0, \quad \forall x \in \overline{\Omega}, y \in \overline{\Omega}^*.$$

$$(A3w) \quad \mathcal{F}(x, p; \xi, \eta) := D_{p_i p_j} A_{kl}(x, p) \xi_i \xi_j \eta_k \eta_l \\ \geq 0 \quad \forall \quad x \in \Omega, p \in \mathbb{R}^n, \xi \perp \eta \in \mathbb{R}^n.$$

Conditions A1 and A2 are the same conditions as in [MTW] but condition A3w is the degenerate form of condition A3 in [MTW]. As will be seen in our examples, we do not necessarily require  $c$  to be defined on all of  $\mathbb{R}^n \times \mathbb{R}^n$  and the vectors  $p$  and  $q$  in conditions A1 and A3w need only lie in the ranges of  $c_x(x, y)$  and  $c_y(x, y)$  on  $\Omega \times \Omega^*$ . Moreover, as done at the outset in [MTW], we may also write

$$(1.19) \quad \mathcal{F}(x, p; \xi, \eta) = (c_{ij,rs} - c^{k',l'} c_{ij,k'} c_{l',rs}) c^{r,k} c^{s,l}(x, y) \xi_i \xi_j \eta_k \eta_l$$

where  $y$  and  $p$  are related through A1. This shows that condition A3w is also symmetric in  $x$  and  $y$ .

In our paper [MTW], we also introduced a notion of convexity of domains with respect to cost functions, namely  $\Omega$  is  $c$ -convex, with respect to  $\Omega^*$ , if the image  $c_y(\cdot, y)(\Omega)$  is convex in  $\mathbb{R}^n$  for each  $y \in \Omega^*$ , while analogously  $\Omega^*$  is  $c^*$ -convex, with respect to  $\Omega$ , if the image  $c_x(x, \cdot)(\Omega^*)$  is convex for each  $x$  in  $\Omega$ . For global regularity we need to strengthen these conditions in the same way that convexity is strengthened to uniform convexity. Namely we define  $\Omega$  to be uniformly  $c$ -convex, with respect to  $\Omega^*$ , if  $\Omega$  is  $c$ -convex, with respect to  $\Omega^*$ ,  $\partial\Omega \in C^2$  and there exists a positive constant  $\delta_0$  such that

$$(1.20) \quad [D_i \gamma_j(x) - c^{l,k} c_{ij,l}(x, y) \gamma_k(x)] \tau_i \tau_j(x) \geq \delta_0$$

for all  $x \in \partial\Omega$ ,  $y \in \Omega^*$ , unit tangent vector  $\tau$  and outer unit normal  $\gamma$ . By pulling back with the mappings  $c_y(\cdot, y)$ , we see that this is equivalent to the condition that the image domains  $c_y(\cdot, y)(\Omega)$  be uniformly convex with respect to  $y \in \Omega^*$ . Similarly we call  $\Omega^*$  uniformly  $c^*$ -convex, with respect to  $\Omega$ , when  $c^*(x, y) = c(y, x)$ . Note that if  $\Omega$  is simply connected with boundary  $\partial\Omega \in C^2$ , then  $\Omega$  is  $c$ -convex if and only if (1.20) holds for  $\delta_0 = 0$ .

We can now formulate our main estimate.

**Theorem 1.1.** *Let  $c$  be a cost function satisfying hypotheses A1, A2, A3w, with respect to bounded  $C^4$  domains  $\Omega, \Omega^* \in \mathbb{R}^n$  which are respectively uniformly  $c$ -convex,  $c^*$ -convex with respect to each other. Let  $\psi$  be a positive function in  $C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ . Then any elliptic solution  $u \in C^3(\overline{\Omega})$  of the second boundary value problem (1.14), (1.7) satisfies the a priori estimate*

$$(1.21) \quad |D^2 u| \leq C,$$

where  $C$  depends on  $c, \psi, \Omega, \Omega^*$  and  $\sup_{\Omega} |u|$ .

As we will indicate later, the smoothness assumption on the solution and the data may be reduced. Further regularity also follows from the theory of linear elliptic equations for example if  $c, \Omega, \Omega^*, \psi$  are  $C^\infty$  then the solution  $u \in C^\infty(\overline{\Omega})$ . The dependence of the estimate (1.21) on  $\sup_\Omega |u|$  may be removed if  $\psi$  is independent of  $u$  as in (1.8).

As a consequence of Theorem 1.1, we may conclude existence theorems for classical solutions by the method of continuity.

**Theorem 1.2.** *Suppose in addition to the above hypotheses that the function  $\psi$  satisfies (1.8) (1.9). Then there exists a unique (up to additive constants) elliptic solution  $u \in C^3(\overline{\Omega})$  of the second boundary value problem (1.14), (1.7).*

From Theorem 1.2, we also obtain an existence result for classical solutions of the Monge-Kantorovich problem in optimal transportation. As above we let  $c \in C^4(\mathbb{R}^n \times \mathbb{R}^n)$  be a cost function and  $\Omega, \Omega^*$  be two bounded domains in  $\mathbb{R}^n$  satisfying the hypotheses of Theorem 1.1. Let  $f > 0, \in C^2(\overline{\Omega})$ ,  $g > 0, \in C^2(\overline{\Omega}^*)$  be positive densities satisfying the mass balance condition (1.9). Then the corresponding optimal transportation problem is to find a measure preserving mapping  $T_0 : \Omega \rightarrow \Omega^*$  which maximizes the cost functional

$$(1.22) \quad C(T) = \int_{\Omega} f(x)c(x, T(x))dx$$

among all measure preserving mappings  $T$  from  $\Omega$  to  $\Omega^*$ . A mapping  $T : \Omega \rightarrow \Omega^*$  is called measure preserving if it is Borel measurable and for any Borel set  $E \subset \Omega^*$ ,

$$(1.23) \quad \int_{T^{-1}(E)} f = \int_E g.$$

**Theorem 1.3.** *Under the above hypotheses, there exists a unique diffeomorphism  $T \in [C^2(\overline{\Omega})]^n$  maximizing the functional (1.22), given by*

$$(1.24) \quad T(x) = Y(x, Du(x)),$$

where  $u$  is an elliptic solution of the boundary problem (1.7), (1.14).

The solution  $u$  of (1.7), (1.14) is called a potential. Note that in [MTW] and elsewhere the cost functions and potentials are the negatives of those here and the optimal transportation problem is written, (in its usual form), as a minimization problem.

The plan of this paper is as follows. In Section 2, we prove that boundary conditions of the form (1.7) are oblique with respect to functions for which the Jacobian  $DT$  is non-singular and we estimate the obliqueness for solutions of the boundary value problem (1.14), (1.7) under hypotheses A1 and A2, (Theorem 2.1). Here the twin assumptions of  $\Omega$  and  $\Omega^*$  being uniformly  $c$  and  $c^*$ -convex with respect to each other are critical.

In Section 3, we prove that second derivatives of solutions of equation (1.14) can be estimated in terms of their boundary values under hypothesis A3w, (Theorem 3.1). This estimation is already immediate from [MTW] when the non-degenerate condition A3 is satisfied. The argument is carried out for equations of the general form (1.1) (with symmetric  $A$ ), in the presence of a global barrier, which is not necessary in the optimal transportation case (Theorem 3.2). This estimation also arises in the treatment of the classical Dirichlet problem [T3]. The proof of the global second derivative estimates in Theorem 1.1 for solutions of the boundary value problem (1.14), (1.7) is completed in Section 4. Here the procedure is similar to that in [LTU] and [U1]. We remark here that this global estimate also extends to the more general prescribed Jacobian equation (1.6) [T3]. In Section 5, we commence the proof of the existence result, Theorem 1.2, by adapting the method of continuity [GT] and establish the result under a stronger uniform  $c$ -convexity hypothesis, (employed in earlier versions of this paper). Section 6 is devoted to the applications to optimal transportation and the proof of Theorem 1.3 from Theorem 1.2, which implies the global regularity of the potential functions in [MTW], under condition A3w. In Section 7, we finally complete the proof of Theorem 1.2 in its full generality, by showing that there exists a smooth function satisfying the ellipticity condition (1.2), together with the boundary condition (1.7), (at least for approximating domains). In the last section, we discuss our results in the light of examples, most of which are already given in [MTW].

## 2. Obliqueness

In this section, we prove that the boundary condition (1.7) implies an oblique boundary condition and estimate the obliqueness. First we recall that a boundary condition of the form

$$(2.1) \quad G(x, u, Du) = 0 \quad \text{on } \partial\Omega$$

for a second order partial differential equation in a domain  $\Omega$  is called oblique if

$$(2.2) \quad G_p \cdot \gamma > 0$$

for all  $(x, z, p) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}^n$ , where  $\gamma$  denotes the unit outer normal to  $\partial\Omega$ . Let us now assume that  $\varphi$  and  $\varphi^*$  are  $C^2$  defining functions for  $\Omega$  and  $\Omega^*$  respectively, with  $\varphi, \varphi^* < 0$  near  $\partial\Omega, \partial\Omega^*$ ,  $\varphi = 0$  on  $\partial\Omega$ ,  $\varphi^* = 0$  on  $\partial\Omega^*$ ,  $\nabla\varphi, \nabla\varphi^* \neq 0$  near  $\partial\Omega, \partial\Omega^*$ . Then if  $u \in C^2(\overline{\Omega})$  is an elliptic solution of the second boundary value problem (1.4), (1.7), we must have

$$(2.2) \quad \varphi^*(Tu) = 0 \quad \text{on } \partial\Omega, \quad \varphi^*(Tu) < 0 \quad \text{near } \partial\Omega.$$

By tangential differentiation, we obtain

$$(2.3) \quad \varphi_i^*(D_j T^i u) \tau_j = 0$$

for all unit tangent vectors  $\tau$ , whence

$$(2.4) \quad \varphi_i^*(D_j T^i) = \chi \gamma_j$$

for some  $\chi \geq 0$ . Consequently

$$(2.5) \quad \varphi_i^* c^{i,k} (u_{jk} - c_{jk}) = \chi \gamma_j,$$

that is

$$(2.6) \quad \varphi_i^* c^{i,k} w_{jk} = \chi \gamma_j,$$

where

$$(2.7) \quad w_{ij} = u_{ij} - c_{ij}.$$

At this point we observe that  $\chi > 0$  on  $\partial\Omega$  since  $|\nabla \varphi^*| \neq 0$  on  $\partial\Omega$  and  $\det DT \neq 0$ . Using the ellipticity of (1.4) and letting  $[w^{ij}]$  denote the inverse matrix of  $[w_{ij}]$ , we then have

$$(2.8) \quad \varphi_i^* c^{i,k} = \chi w^{jk} \gamma_j.$$

Now writing

$$(2.9) \quad G(x, p) = \varphi^*(Y(x, p)),$$

we have

$$(2.10) \quad \beta_k := G_{p_k}(\cdot, Du) = \chi w^{jk} \gamma_j,$$

whence

$$(2.11) \quad \begin{aligned} \beta \cdot \gamma &= \chi w^{ij} \gamma_i \gamma_j \\ &> 0 \end{aligned}$$

on  $\partial\Omega$ . We obtain a further formula for  $\beta \cdot \gamma$ , from (2.6), namely

$$(2.12) \quad \begin{aligned} \varphi_i^* c^{i,k} w_{jk} \varphi_l^* c^{l,j} &= \chi \varphi_l^* c^{l,j} \gamma_j \\ &= \chi (\beta \cdot \gamma). \end{aligned}$$

Eliminating  $\chi$  from (2.11) and (2.12), we have

$$(2.13) \quad (\beta \cdot \gamma)^2 = (w^{ij} \gamma_i \gamma_j) (w_{kl} c^{i,k} c^{j,l} \varphi_i^* \varphi_j^*).$$

We call (2.13) a formula of Urbas type, as it was proved by Urbas [U1] for the special case,  $c(x, y) = x \cdot y$ ,  $Y(\xi, p) = p$ , of the Monge-Ampère equation. Note that to prove (2.13), we only used conditions A1 and A2 and moreover (2.13) continues to hold in the generality of (1.4).

Our main task now is to estimate  $\beta \cdot \gamma$  from below for solutions of (1.14), (1.7). For this in addition to conditions A1, A2, we also need the uniform  $c$  and  $c^*$  convexity of  $\Omega$  and  $\Omega^*$  respectively. Our approach is similar to [U1] for the special case of the Monge-Ampère equation and begins by invoking the key idea from [T] for estimating double normal derivatives of solution of the Dirichlet problem. Namely we fix a point  $x_0$  on  $\partial\Omega$  where  $\beta \cdot \gamma$  is minimized, for an elliptic solution  $u \in C^3(\overline{\Omega})$ , and use a comparison argument to estimate  $\gamma \cdot D(\beta \cdot \gamma)$  from above. Without some concavity condition in  $p$  the quantity  $\beta \cdot \gamma$  does not satisfy a nice differential inequality so we will get around this by considering instead the function

$$(2.14) \quad v = \beta \cdot \gamma - \kappa \varphi^*(Tu)$$

for sufficiently large  $\kappa$ , where now the defining function  $\varphi^*$  is chosen so that

$$(2.15) \quad (D_{ij} \varphi^*(Tu) - c^{k,l} c_{l,ij}(\cdot, Tu) D_k \varphi^*(Tu) \xi_i \xi_j \geq \delta_0^* |\xi|^2$$

near  $\partial\Omega$ , for all  $\xi \in \mathbb{R}^n$  and some positive constant  $\delta_0^*$ . Inequality (2.15) is possible by virtue of the uniform  $c^*$ -convexity of  $\Omega^*$ , with the function  $\varphi^*$  given, for example by

$$(2.16) \quad \varphi^* = -ad^* + b(d^*)^2,$$

where  $a$  and  $b$  are positive constants and  $d^*$  denotes the distance function for  $\Omega^*$ , [GT].

By differentiation of equation (1.14), in the form (1.1), we obtain, for  $r = 1, \dots, n$ ,

$$(2.17) \quad w^{ij} \{D_{ij} u_r - D_{p_k} A_{ij}(x, Du) D_k u_r - D_{x_r} A_{ij}(x, Du)\} = D_r \log B.$$

Introducing the linearized operator  $L$ ,

$$(2.18) \quad Lv = w^{ij} (D_{ij} v - D_{p_k} A_{ij} D_k v),$$

we need to compute  $Lv$  for  $v$  given by (2.14). Setting

$$(2.19) \quad F(x, p) = G_p(x, p) \cdot \gamma(x) - \kappa G(x, p),$$



where  $G$  is defined by (2.9), we see that

$$(2.20) \quad v(x) = F(x, Du(x)).$$

Writing  $b_k^{ij} = -D_{p_k} A_{ij}$ , we then have

$$(2.21) \quad \begin{aligned} Lv = & w^{ij} \{ F_{p_r} D_{ij} u_r + F_{p_r p_s} D_{ir} u D_{js} u \\ & + F_{x_i x_j} + 2F_{x_i p_r} D_{jr} u + b_k^{ij} (F_{x_k} + F_{p_r} D_k u_r) \} \end{aligned}$$

In the ensuing calculations, we will often employ the following formulae,

$$(2.22) \quad \begin{aligned} c_k^{i,j}(x, y) &= D_{x_k} c^{i,j}(x, y) \\ &= -c^{i,l} c^{r,j} c_{kl,r}(x, y), \\ c_{,k}^{i,j}(x, y) &= D_{y_k} c^{i,j}(x, y) \\ &= -c^{i,l} c^{r,j} c_{l,kr}(x, y), \end{aligned}$$

as well as (1.13). Indeed, using (1.13) and (2.22), we have

$$(2.23) \quad \begin{aligned} G_{p_i p_j} &= D_{p_j} (\varphi_k^* c^{k,i}) \\ &= \varphi_{kl}^* c^{k,i} c^{l,j} - \varphi_k^* c^{s,j} c_{,s}^{k,i} \\ &= c^{k,i} c^{l,j} \{ \varphi_{kl}^* - \varphi_r^* c^{r,s} c_{s,kl} \} \end{aligned}$$

so that

$$(2.24) \quad \begin{aligned} G_{p_i p_j}(x, Du) \xi_i \xi_j &\geq \delta_0^* \sum |c^{i,j} \xi_j|^2 \\ &\geq \kappa_0^* |\xi|^2 \end{aligned}$$

for a further positive constant  $\kappa_0^*$ . By choosing  $\kappa$  sufficiently large, we can then ensure that

$$(2.25) \quad F_{p_i p_j}(x, Du) \xi_i \xi_j \leq -\frac{1}{2} \kappa |\xi|^2$$

near  $\partial\Omega$ . Substituting into (2.20) and using (2.16), it follows that

$$(2.26) \quad Lv \leq -\frac{1}{4} \kappa w_{ii} + C(w^{ii} + 1) + D_{p_k} \log B D_k v,$$

where  $C$  is a constant depending on  $c, \psi, \Omega$  and  $\Omega^*$ , as well as  $\kappa$ .

Next we observe that unless the defining function  $\varphi^*$  extends to all of  $\Omega^*$  so that (2.15) is satisfied for all  $Tu \in \Omega^*$ , we have no control on the neighbourhood of  $\partial\Omega$ , where (2.26)

holds. This is remedied by replacing  $G$  in (2.19) by a function satisfying (2.24) in all of  $\Omega$ , agreeing with (2.9) near  $\partial\Omega$ , for example by taking

$$(2.27) \quad G(x, p) = m_h\{(\varphi^*(Y(x, p)), a_1(|p|^2 - K^2)\},$$

where  $a_1$  and  $K$  are positive constants, with  $a_1$  sufficiently small and  $K > \max|Du|$ , and for  $h$  sufficiently small,  $m_h$  is the mollification of the max-function of two variables.

A suitable barrier is now provided by the uniform  $c$ -convexity of  $\Omega$  which implies, analogously to the case of  $\Omega^*$  above, that there exists a defining function  $\varphi$  for  $\Omega$  satisfying

$$(2.28) \quad [D_{ij}\varphi - c^{l,k}c_{ij,l}(x, Tu)D_k\varphi]\xi_i\xi_j \geq \delta_0|\xi|^2,$$

in a fixed neighbourhood of  $\partial\Omega$ , (for a constant  $\delta_0 > 0$ ). By appropriate choice, of say the constants  $a$  and  $b$  in (2.16), without the  $*$ , (or following the uniformly convex case in [GT, Chapter 14]), we may obtain, by virtue of (2.21),

$$(2.29) \quad L\varphi \geq \delta_0 w^{ii} + Kw^{ij}D_i\varphi D_j\varphi,$$

for a given constant  $K$ . Combining (2.26) and (2.29), and using the positivity of  $B$ , we then infer by the usual barrier argument, (which entails fixing a small enough neighbourhood of  $\partial\Omega$ , [GT]),

$$(2.30) \quad \gamma \cdot Dv(x_0) \leq C,$$

where again  $C$  is a constant depending on  $c, \Omega, \Omega^*$  and  $\psi$ . From (2.28) and since  $x_0$  is a minimum point of  $v$  on  $\partial\Omega$ , we can write

$$(2.31) \quad Dv(x_0) = \tau\gamma(x_0)$$

where  $\tau \leq C$ . To use the information embodied in (2.31), we need to calculate

$$(2.32) \quad \begin{aligned} D_i(\beta \cdot \gamma) &= D_i\{\varphi_k^* c^{k,j}\gamma_j\} \\ &= \varphi_{kl}^* D_i(T^l u) c^{k,j}\gamma_j + \varphi_k^* c^{k,j} D_i\gamma_j + \varphi_k^* \gamma_j (c_i^{k,j} + c_l^{k,j} D_i T^l u) \\ &= \varphi_k^* c^{k,j} (D_i\gamma_j - c^{s,r} c_{ij,s} \gamma_r) + (\varphi_{kl}^* - \varphi_r^* c^{r,s} c_{s,kl}) c^{k,j} \gamma_j D_i T^l u \end{aligned}$$

Multiplying by  $\varphi_t^* c^{t,i}$  and summing over  $i$ , we obtain

$$(2.33) \quad \begin{aligned} \varphi_t^* c^{t,i} D_i(\beta \cdot \gamma) &= \varphi_k^* \varphi_t^* c^{k,j} c^{t,i} (D_i\gamma_j - c^{s,r} c_{ij,s} \gamma_r) \\ &\quad + \varphi_t^* c^{t,i} c^{k,j} \gamma_j c^{l,m} w_{im} (\varphi_{kl}^* - \varphi_r^* c^{r,s} c_{s,kl}) \\ &\geq \delta_0 \sum |\varphi_i^* c^{i,j}|^2 \end{aligned}$$

by virtue of the uniform  $c$ -convexity of  $\Omega$ , the  $c^*$ -convexity of  $\Omega^*$  and (2.6). Consequently, from (2.19) and (2.31), we obtain at  $x_0$ ,

$$(2.34) \quad -\kappa w_{kl} c^{i,k} c^{j,l} \varphi_i^* \varphi_j^* \leq C(\beta \cdot \gamma) - \tau_0$$

for positive constants,  $C$  and  $\tau_0$ . Hence if  $\beta \cdot \gamma \leq \tau_0/2C$ , we have the lower bound

$$(2.35) \quad w_{kl} c^{i,k} c^{j,l} \varphi_i^* \varphi_j^* \geq \frac{\tau_0}{2\kappa}.$$

To complete the estimation of  $\beta \cdot \gamma$  we invoke the dual problem to estimate  $w^{ij} \gamma_i \gamma_j$  at  $x_0$ . Assuming for the moment that  $Tu$  is one to one, we let  $u^*$  denote the  $c$ -transform of  $u$ , defined for  $y = Tu(x) \in \Omega^*$  by

$$(2.36) \quad u^*(y) = c(x, y) - u(x).$$

It follows that

$$(2.37) \quad \begin{aligned} Du^*(y) &= c_y(x, y) \\ &= c_y(T^*u^*(y), y), \end{aligned}$$

where

$$(2.38) \quad \begin{aligned} T^*u^*(y) &= X(Du^*, y) \\ &= (Tu)^{-1}(y), \end{aligned}$$

and the second boundary value problem (1.14), (1.7) is equivalent to

$$(2.39) \quad |\det D_y(T^*u^*)| = g(y)/f(T^*u^*) \quad \text{in } \Omega^*,$$

$$(2.40) \quad T^*\Omega^* = \Omega.$$

Noting that the defining functions  $\varphi$  and  $\varphi^*$  may be chosen so that  $\nabla \varphi = \gamma$ ,  $\nabla \varphi^* = \gamma^*$  on  $\partial\Omega$ ,  $\partial\Omega^*$  respectively, we clearly have for  $x \in \partial\Omega$ ,  $y \in Tu(x) \in \partial\Omega^*$ ,

$$(2.41) \quad \begin{aligned} \beta \cdot \gamma(x) &= c^{k,i}(x, y) \varphi_i \varphi_k^*(y) \\ &= \beta^* \cdot \gamma^*(y), \end{aligned}$$

where

$$(2.42) \quad \beta^*(y) = D_q \varphi(Y^*(D_y u^*, y)).$$

Hence the quantity  $\beta^* \cdot \gamma^*$  is minimized on  $\partial\Omega^*$  at the point  $y_0 = Tu(x_0)$ . Furthermore, for  $y = Tu(x)$ ,  $x \in \partial\Omega$ ,

$$(2.43) \quad w^{ij} \gamma_i \gamma_j(x) = w_{kl}^*(y) c^{k,i} c^{l,j}(x, y) \varphi_i \varphi_j(x),$$

where

$$(2.44) \quad w_{kl}^*(y) = u_{y_k y_l}^*(y) - c_{,kl}(x, y).$$

Applying now the estimate (2.35) to  $u^*$  at the point  $y_0 \in \partial\Omega^*$ , we finally conclude from (2.13) the desired obliqueness estimate

$$(2.45) \quad \beta \cdot \gamma \geq \delta$$

on  $\partial\Omega$  for some positive constant  $\delta$  depending only on  $\Omega, \Omega^*, c$ , and  $\psi$ .

The above argument clearly extends to arbitrary positive terms  $B$  (1.15). Noting also that it suffices in the above argument that  $T$  need only be one-to-one from a neighbourhood of the point  $x_0$  to a neighbourhood of  $y_0$ , we have the following theorem.

**Theorem 2.1.** *Let  $c \in C^3(\mathbb{R}^n \times \mathbb{R}^n)$  be a cost function satisfying hypotheses A1, A2, with respect to bounded  $C^3$  domains  $\Omega, \Omega^* \subset \mathbb{R}^n$ , which are respectively uniformly  $c$ -convex,  $c^*$ -convex with respect to each other. Let  $\psi$  be a positive function in  $C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ . Then any elliptic solution  $u \in C^3(\overline{\Omega})$ , of the second boundary value problem (1.14), (1.7) satisfies the obliqueness estimate (2.45).*

Note that  $Tu$  is automatically globally one-to-one under the hypotheses of Theorem 1.2 by virtue of the change of variables formula. In ensuing work, (see [T3]), we extend Theorem 2.1 to the more general prescribed Jacobian equation (1.4). The main difference is that we cannot directly use the  $c$ -transform to get the complementary estimate to (2.35). Instead the quantities there are transformed using the local diffeomorphism  $Tu$ . Indeed we could also have avoided the use of duality in the proof of Theorem 2.1 by direct transformation of (2.31).

### 3. Global second derivative bounds

In this section we show that the second derivatives of elliptic solutions of equation (1.14) may be estimated in terms of their boundary values. For this estimation and the boundary estimates in the next section, it suffices to consider the general form (1.1) under the assumption that the matrix valued function  $A \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n)$  satisfies condition A3w, that is

$$(3.1) \quad D_{p_k p_l} A_{ij}(x, z, p) \xi_i \xi_j \eta_k \eta_l \geq 0$$

for all  $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ ,  $\xi, \eta \in \mathbb{R}^n$ ,  $\xi \perp \eta$ . We also assume  $A$  is symmetric, which is the case for the optimal transportation equation (1.14). When (3.1) is strengthened to the condition A3 in [MTW], that is

$$(3.2) \quad D_{p_k p_l} A_{ij}(x, z, p) \xi_i \xi_j \eta_k \eta_l \geq \delta |\xi|^2 |\eta|^2$$

for some constant  $\delta > 0$ , for all  $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ ,  $\xi, \eta \in \mathbb{R}^n$ ,  $\xi \perp \eta$ , then the global second derivative estimate follows immediately from our derivation of interior estimates in [MTW]. In the general case the proof is much more complicated and we need to also assume some kind of barrier condition, namely that there exists a function  $\tilde{\varphi} \in C^2(\overline{\Omega})$  satisfying

$$(3.3) \quad [D_{ij} \tilde{\varphi}(x) - D_{p_k} A_{ij}(x, z, p) D_k \tilde{\varphi}(x)] \xi_i \xi_j \geq \tilde{\delta} |\xi|^2$$

for some positive  $\tilde{\delta} > 0$  and for all  $\xi \in \mathbb{R}^n$ ,  $x, z, p \in$  some set  $U \subset \Omega \times \mathbb{R} \times \mathbb{R}^n$ , whose projection on  $\Omega$  is  $\Omega$ . In general, condition (3.1) implies some restriction on the domain  $\Omega$ , but for the case of equations arising in optimal transportation, it can be avoided by a duality argument.

Our reduction to the boundary estimation follows the approach in [GT], originating with Pogorelov, with some modification analogous to that in [LTU]. Let  $u \in C^4(\Omega)$  be an elliptic solution of equation (1.1), with  $x, u(x), Du(x) \in U$  for  $x \in \Omega$  and  $\xi$  a unit vector in  $\mathbb{R}^n$ . Let  $v$  be the auxiliary function given by

$$(3.4) \quad v = v(\cdot, \xi) = \log(w_{ij} \xi_i \xi_j) + \tau |Du|^2 + \kappa \tilde{\varphi},$$

where  $w_{ij} = D_{ij} u - A_{ij}$ . By differentiation of equation (1.1), we have

$$(3.5) \quad \begin{aligned} w^{ij} [D_{ij} u_\xi - D_\xi A_{ij} - (D_z A_{ij}) u_\xi - (D_{p_k} A_{ij}) D_k u_\xi] \\ = D_\xi \tilde{B} + (D_z \tilde{B}) u_\xi + (D_{p_k} \tilde{B}) D_k u_\xi, \end{aligned}$$

where  $\tilde{B} = \log B$ . A further differentiation yields

$$(3.6) \quad \begin{aligned} w^{ij} [D_{ij} u_{\xi\xi} - D_{\xi\xi} A_{ij} - (D_{zz} A_{ij})(u_\xi)^2 - (D_{p_k p_l} A_{ij}) D_k u_\xi D_l u_\xi \\ - (D_z A_{ij}) u_{\xi\xi} - (D_{p_k} A_{ij}) D_k u_{\xi\xi} - 2(D_{\xi z} A_{ij}) u_\xi \\ - 2(D_{\xi p_k} A_{ij}) D_k u_\xi - 2(D_{z p_k} A_{ij})(D_k u_\xi) u_\xi] - w^{ik} w^{jl} D_\xi w_{ij} D_\xi w_{kl} \\ = D_{\xi\xi} \tilde{B} + (D_{zz} \tilde{B}) u_\xi^2 + (D_{p_k p_l} \tilde{B}) D_k u_\xi D_l u_\xi \\ + 2(D_{\xi z} \tilde{B}) u_\xi + 2(D_{\xi p_k} \tilde{B}) D_k u_\xi + 2(D_{z p_k} \tilde{B})(D_k u_\xi) u_\xi \\ + (D_z \tilde{B}) u_{\xi\xi} + (D_{p_k} \tilde{B}) D_k u_{\xi\xi}. \end{aligned}$$

Furthermore differentiating (3.4) we have

(3.7)

$$D_i v = \frac{D_i w_{\xi\xi}}{w_{\xi\xi}} + 2\tau D_k u D_{ki} u + \kappa D_i \tilde{\varphi},$$

(3.8)

$$D_{ij} v = \frac{D_{ij} w_{\xi\xi}}{w_{\xi\xi}} - \frac{D_i w_{\xi\xi} D_j w_{\xi\xi}}{w_{\xi\xi}^2} + 2\tau (D_{ik} u D_{jk} u + D_k u D_{ijk} u) + \kappa D_{ij} \tilde{\varphi}.$$

where we have written  $w_{\xi\xi} = D_{ij} w_{\xi_i \xi_j}$ . Using condition A3w in (3.6) and retaining all terms involving third derivatives, we estimate

(3.9)

$$\begin{aligned} Lu_{\xi\xi} &:= w^{ij} (D_{ij} u_{\xi\xi} + b_k^{ij} D_k u_{\xi\xi}) - (D_{p_k} \tilde{B}) D_k u_{\xi\xi} \\ &\geq w^{ik} w^{jl} D_\xi w_{ij} D_\xi w_{kl} - C\{(1 + w_{ii})w^{ii} + (w_{ii})^2\} \end{aligned}$$

where, as in the previous section,  $b_k^{ij} = -D_{p_k} A_{ij}$  and  $C$  is a constant depending on the first and second derivatives of  $A$  and  $\log B$  and  $\sup_\Omega (|u| + |Du|)$ . To apply A3w, we fix a point  $x \in \Omega$  and choose coordinate vectors as the eigenfunctions of the matrix  $[w_{ij}]$  corresponding to eigenvalues  $0 < \lambda_1 \leq \dots \leq \lambda_n$ . Writing  $A_{ij,kl} = D_{p_k p_l} A_{ij}$ , we then estimate

$$\begin{aligned} w^{ij} A_{ij,kl} u_{k\xi} u_{l\xi} &\geq w^{ij} A_{ij,kl} w_{k\xi} w_{l\xi} - C w^{ii} (1 + w_{ii}) \\ &\geq \sum_{k \text{ or } l=r} \frac{1}{\lambda_r} A_{rr,kl} (\lambda_k \xi_k) (\lambda_l \xi_l) - C w^{ii} (1 + w_{ii}) \\ &\geq -C\{w^{ii} (1 + w_{ii}) + w_{ii}\} \end{aligned}$$

From (3.9), we obtain also

(3.10)

$$Lw_{\xi\xi} \geq w^{ik} w^{jl} D_\xi w_{ij} D_\xi w_{kl} - C\{(1 + w_{ii})w^{ii} + w_{ii}^2\}$$

for a further constant  $C$ . Here we use equation (3.5) to control the third derivative term arising from differentiating  $A_{kl} \xi_k \xi_l$ . From (3.8) and (3.10), we obtain, after some reduction,

(3.11)

$$\begin{aligned} Lv &\geq \frac{1}{w_{\xi\xi}} w^{ik} w^{jl} D_\xi w_{ij} D_\xi w_{kl} - \frac{1}{w_{\xi\xi}^2} w^{ij} D_i w_{\xi\xi} D_j w_{\xi\xi} \\ &\quad + 2\tau w_{ii} + \kappa w^{ii} - C\left\{\frac{1}{w_{\xi\xi}} [(1 + w_{ii})w^{ii} + w_{ii}^2] + \tau + \kappa\right\} \end{aligned}$$

Now suppose  $v$  takes its maximum at a point  $x_0 \in \Omega$  and a vector  $\xi$ , which we take to be  $e_1$ . We need to control the first two terms on the right hand side of (3.11). To do this

we choose remaining coordinates so that  $[w_{ij}]$  is diagonal at  $x_0$ . Then we estimate

$$\begin{aligned}
(3.12) \quad & \frac{1}{w_{\xi\xi}} w^{ik} w^{jl} D_\xi w_{ij} D_\xi w_{kl} - \frac{1}{w_{\xi\xi}^2} w^{ij} D_i w_{\xi\xi} D_j w_{\xi\xi} \\
&= \frac{1}{w_{11}} w^{ii} w^{jj} (D_1 w_{ij})^2 - \frac{1}{w_{11}^2} w^{ii} (D_i w_{11})^2 \\
&\geq \frac{1}{w_{11}^2} \sum_{i>1} [2w^{ii} (D_1 w_{1i}^2 - w^{ii} (D_i w_{11})^2)] \\
&= \frac{1}{w_{11}^2} \sum_{i>1} w^{ii} (D_i w_{11})^2 \\
&\quad + \frac{2}{w_{11}^2} \sum_{i>1} w^{ii} [D_1 w_{1i} - D_i w_{11}] [D_1 w_{1i} + D_i w_{11}] \\
&\geq \frac{1}{w_{11}^2} \sum_{i>1} w^{ii} (D_i w_{11})^2 \\
&\quad + \frac{2}{w_{11}^2} \sum_{i>1} w^{ii} [D_i A_{11} - D_1 A_{1i}] [2D_i w_{11} + D_i A_{11} - D_1 A_{1i}] \\
&\geq -C w^{ii}
\end{aligned}$$

Combining (3.11) with (3.12), we obtain the estimate, at  $x_0$ ,

$$(3.13) \quad Lv \geq \tau w_{ii} + \kappa w^{ii} - C\{\tau + \kappa\},$$

for either  $\tau$  or  $\kappa$  sufficiently large. Note that when we use (3.7) in the second last line of (3.12), we improve (3.13) by retention of the term  $\sum_{i>1} w^{ii} (D_i w_{11})^2$  on the right hand side, which corresponds to the key term in the Pogorelov argument for interior estimates [GT].

From (3.13), we finally obtain an estimate from above for  $w_{ii}(x_0)$ , which we formulate in the following theorem.

**Theorem 3.1.** *Let  $u \in C^4(\Omega)$  be an elliptic solution of equation (1.1) in  $\Omega$ , with  $x, u(x), Du(x) \in U$ , for all  $x \in \Omega$ . Suppose the conditions A3w and (3.3) hold and  $B$  is a positive function in  $C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ . Then we have the estimate*

$$(3.14) \quad \sup_{\Omega} |D^2 u| \leq C(1 + \sup_{\partial\Omega} |D^2 u|),$$

where the constant  $C$  depends on  $A, B, \Omega, \Omega^*$  and  $\sup_U(|z| + |p|)$ .

Note that we only need the condition A3w to hold on the set  $U$ .

From the proof of Theorem 3.1 we obtain the corresponding estimate for equation (1.14), without the barrier condition (3.3).

**Theorem 3.2.** *Let  $u \in C^4(\Omega)$  be an elliptic solution of equation (1.14) in  $\Omega$  with  $Tu(\Omega) \subset \Omega^*$ . Suppose the cost function  $c$  satisfies hypotheses A1, A2, A3w and  $B$  is a positive function in  $C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ . Then we have the estimate (3.14).*

To prove Theorem 3.2, we take  $\kappa = 0$  in the proof of Theorem 3.1, to obtain an estimate for  $w_{ii}$  in terms of  $w^{ii}$ , that is

$$(3.15) \quad w_{ii} \leq \varepsilon \sup_{\Omega} w^{ii} + C_{\varepsilon} \left(1 + \sup_{\partial\Omega} |D^2 u|\right),$$

for arbitrary  $\varepsilon > 0$ , with constant  $C_{\varepsilon}$  also depending on  $\varepsilon$ . If  $T$  is globally one-to-one, we then conclude (3.13), in the optimal transportation case, by using the dual problem (2.37), (2.38). More generally, we consider the dual function  $v^*$  in place of (3.4), given by

$$(3.16) \quad v^* = v(x, \xi) = \log(w^{ij} c_{i,k} c_{j,l} \xi_k \xi_l) + \tau |c_y(x, Tu(x))|^2$$

and suppose it is maximized at a point  $x_0^*$  in  $\Omega$ . Since  $T$  will now be one-to-one from a neighborhood  $\mathcal{N}$  of  $x_0^*$  to a neighbourhood  $\mathcal{N}^*$  of  $y_0^* = Tu(x_0^*)$ , we may then proceed as before, noting that in  $\mathcal{N}^*$ ,  $v^*$  is given by (3.4) with  $u$  replaced by its  $c$ -transform  $u^*$ .

The estimate (3.14) arose from our investigation of the classical Dirichlet problem, (see [T3]). We remark also that from (3.16), we see that (3.3) is also not needed when  $n = 2$ .

#### 4. Boundary estimates for the second derivatives

This part of our argument is similar to the treatment of the oblique boundary value problems for Monge-Ampère equations in [LTU, U3]. The paper [LTU] concerned the Neumann problem, utilizing a delicate argument which did not extend to other linear oblique boundary conditions. For nonlinear oblique conditions of the form (2.1) where the function  $G$  is uniformly convex in the gradient, the twice tangential differentiation of (2.1) yields quadratic terms in second derivatives which compensate for the deviation of  $\beta = G_p$  from the geometric normal and permit some technical simplification for general inhomogeneous terms  $\psi$  [U3].

First we deal with the non-tangential second derivatives. Letting  $F \in C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$  and  $v = F(\cdot, u, Du)$ , where  $u \in C^3(\overline{\Omega})$  is an elliptic solution of equation (1.1), we have from our calculation in Section 2,

$$(4.1) \quad |Lv| \leq C(w^{ii} + w_{ii} + 1),$$



where  $L$  is given by (2.17) and  $C$  is a constant depending on  $A, B, G, \Omega$  and  $|u|_{1;\Omega}$ . Now using the equation (1.1) itself, we may estimate

$$(4.2) \quad w_{ii}^{\frac{1}{n-1}} \leq Cw^{ii},$$

so that, writing  $M = \sup_{\Omega} w_{ii}$ , we have from (4.1)

$$(4.3) \quad |Lv| \leq C(1 + M)^{\frac{n-2}{n-1}} w^{ii}.$$

Hence, if there exists a  $C^2$  defining function  $\varphi$  satisfying (3.3) near  $\partial\Omega$ , together with  $\varphi = 0$  on  $\partial\Omega$ , we obtain by the usual barrier argument, taking  $F = G$ ,

$$(4.4) \quad |D(\beta \cdot Du)| \leq C(1 + M)^{\frac{n-2}{n-1}}$$

on  $\partial\Omega$ , so that in particular

$$(4.5) \quad w_{\beta\beta} \leq C(1 + M)^{\frac{n-2}{n-1}}$$

on  $\partial\Omega$ . Now for any vector  $\xi \in \mathbb{R}^n$ , we have

$$(4.6) \quad w_{\xi\xi} = w_{\tau\tau} + b(w_{\tau\beta} + w_{\beta\tau}) + b^2 w_{\beta\beta},$$

where

$$(4.7) \quad b = \frac{\xi \cdot \gamma}{\beta \cdot \gamma}, \quad \tau = \xi - b\beta.$$

Suppose  $w_{\xi\xi}$  takes its maximum over  $\partial\Omega$  and tangential  $\xi$ ,  $|\xi| = 1$  at  $x_0 \in \partial\Omega$  and  $\xi = e_1$ . Then from (4.5) and (4.6) and tangential differentiation of the boundary condition (2.1) we have on  $\partial\Omega$ ,

$$(4.8) \quad w_{11} \leq |e_1 - b\beta|^2 w_{11}(0) + bF(\cdot, u, Du) + Cb^2(1 + M)^{\frac{n-2}{n-1}},$$

for a given function  $F \in C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ . Combining (2.26), (3.10), (4.1) and (4.2) and utilizing a similar barrier argument to that in Section 2, we thus obtain the third derivative estimate

$$(4.9) \quad -D_{\beta} w_{11}(x_0) \leq C(1 + M)^{\frac{2n-3}{n-1}},$$

Differentiating (2.1) twice in a tangential direction  $\tau$ , with  $\tau(x_0) = e_1$ , we obtain at  $x_0$ ,

$$(4.10) \quad (D_{p_k p_l} G) u_{1k} u_{1l} + (D_{p_k} G) u_{11k} \leq C(1 + M),$$

whence we conclude from (4.9)

$$(4.11) \quad \max_{\partial\Omega} |D^2 u| \leq C(1 + \sup_{\Omega} |D^2 u|)^{\frac{2n-3}{2(n-1)}}$$

by virtue of the uniform convexity of  $G$  with respect to  $p$ . Taking account of the global estimate (3.13), we complete the proof of Theorem 1.1.  $\square$

Once the second derivatives are bounded, the equation (1.1) is effectively uniformly elliptic so that from the obliqueness estimate (2.43), we obtain global  $C^{2,\alpha}$  estimates from the theory of oblique boundary value problems for uniformly elliptic equations in [LT]. By the theory of linear elliptic equations with oblique boundary conditions [GT], we then infer estimates in  $C^{3,\alpha}(\overline{\Omega})$  for any  $\alpha < 1$  from the assumed smoothness of our data. We may also have assumed that our solution  $u \in C^2(\overline{\Omega})$ .

As in the previous section, the technicalities are simpler when condition A3w is strengthened to condition A3 and we also obtain local boundary estimates for the second derivatives. To see this we estimate the tangential second derivatives first by differentiating the equation (1.1) and boundary condition (2.1) twice with respect to a tangential vector field  $\tau$  near a point  $y \in \partial\Omega$ . We then obtain an estimate for  $\eta D_{\tau\tau} u$ , for an appropriately chosen cut-off function  $\eta$ . The mixed tangential-normal second derivatives  $D_{\tau n} u$  are estimated as above by a single tangential differentiation of (2.1) so that the double normal derivative may be obtained either from (4.5) or from the equation (1.1) itself and the estimates in Section 2 for  $w^{ij}\gamma_i\gamma_j$  from below, similarly to the Dirichlet problem, see [T1].

## 5. Method of continuity

To complete the proof of Theorem 1.2, we adapt the method of continuity for nonlinear oblique boundary value problems, presented in [GT] and already used in the special case (1.16) (1.17) [U1]. The situation here is more complicated unless we know in advance that there exists a smooth function  $u_0$ , satisfying the ellipticity condition (1.2) together with the boundary condition (1.7). Later in Section 7, we shall prove the existence of such a function, (at least for approximating domains). Otherwise we need to consider families of subdomains. To commence the procedure, we fix a point  $x_0 \in \Omega$ . Then for sufficiently small radius  $r > 0$ , the ball  $\Omega_0 = B_r(x_0) \subset \Omega$  will be uniformly  $c$ -convex with respect to  $\Omega^*$  and the function  $u_0$ , given by

$$(5.1) \quad u_0(x) = \frac{\kappa}{2}|x - x_0|^2 + p_0 \cdot (x - x_0),$$

will satisfy the ellipticity condition (1.2). Moreover the image  $\Omega_0^* = Tu_0(\Omega_0)$  will be uniformly  $c^*$ -convex with respect to  $\Omega$  with  $Tu_0$  a diffeomorphism from  $\Omega_0$  to  $\Omega_0^*$ . To

see this we observe that

$$(5.2) \quad c_x(x_0, \Omega_0^*) = B_{\kappa r}(p_0),$$

so that by taking  $\kappa r$  small enough, we can fulfill condition (1.19) on  $\partial\Omega_0^*$ , with respect to  $x_0 \in \Omega$ , for constant  $\delta_0 = \frac{1}{\kappa r}$  as large as we wish. Suppose now we can foliate  $\Omega - \Omega_0$  and  $\Omega^* - \Omega_0^*$  by boundaries of  $c$ -convex and  $c^*$ -convex domains, respectively. That is there exist increasing families of domains  $\{\Omega_t\}, \{\Omega_t^*\}$ ,  $0 \leq t \leq 1$ , continuously depending on the parameter  $t$ , such that

- (i)  $\Omega_t \subset \Omega$ ,  $\Omega_t^* \subset \Omega^*$ ,
- (ii)  $\Omega_1 = \Omega$ ,  $\Omega_1^* = \Omega^*$ ,
- (iii)  $\partial\Omega_t, \partial\Omega_t^* \in C^4$ , uniformly with respect to  $t$ ,
- (iv)  $\Omega_t, \Omega_t^*$  are uniformly  $c$ -convex,  $c^*$ -convex with respect to  $\Omega^*, \Omega$ , respectively.

The construction of such a family is discussed below.

Given our families of domains  $\Omega_t, \Omega_t^*$ ,  $0 < t \leq 1$ , we need to define corresponding equations. Let  $B$  be a positive function in  $C^2(\overline{\Omega} \times \mathbb{R}^n)$  and  $f$  a positive function in  $C^2(\overline{\Omega})$  such that

$$(5.3) \quad f = -\sigma u_0 + \log[\det\{D^2u_0 - c_{xx}(\cdot, Y(\cdot, Du_0))\}/B(\cdot, Du_0)]$$

in  $\Omega_0$ , for some fixed constant  $\sigma > 0$ . We then consider the family of boundary value problems:

$$(5.4) \quad \begin{aligned} F[u] &:= \det\{D^2u - c_{xx}(\cdot, Y(\cdot, Du))\} = e^{\sigma u + (1-t)f} B(\cdot, Du), \\ Tu(\Omega_t) &= Y(\cdot, Du)(\Omega_t) = \Omega_t^*, \end{aligned}$$

From our construction and the obliqueness, we see that  $u_0$  is the unique elliptic solution of (5.4) at  $t = 0$ .

From Section 2, we also see that the boundary condition in (5.4) is equivalent to the oblique condition

$$(5.5) \quad G_t(\cdot, Du) := \varphi_t^*(Y(\cdot, Du)) = 0 \quad \text{on } \partial\Omega_t.$$

To adapt the method of continuity from [GT], we fix  $\alpha \in (0, 1)$  and let  $\Sigma$  denote the subset of  $[0, 1]$  for which the problem (5.4) is solvable for an elliptic solution  $u = u_t \in C^{2,\alpha}(\overline{\Omega}_t)$ , with  $Tu$  invertible. We then need to show that  $\Sigma$  is both closed and open in  $[0, 1]$ . First we note that the boundary condition (5.4) implies a uniform bound for  $Du_t$ . Integrating the equation (5.4), we then obtain uniform bounds for the quantities

$$\int_{\Omega_t} e^{\sigma u_t},$$

so that the solutions  $u_t$  will be uniformly bounded for  $\sigma > 0$ . Uniform estimates in  $C^{2,1}(\overline{\Omega})$  now follow from our a priori estimates in Section 4, which are also clearly independent of  $t \in [0, 1]$ . By compactness, we then infer that  $\Sigma$  is closed. To show  $\Sigma$  is open, we use the implicit function theorem and the linear theory of oblique boundary value problems, as in [GT]. The varying domains  $\{\Omega_t\}$  may be handled by means of diffeomorphisms approximating the identity, which transfer the problem (5.4) for  $t$  close to some  $t_0 \in \Sigma$  to a problem in  $\Omega_{t_0}$ . We then conclude the solvability of (5.4) for all  $t \in [0, 1]$ , which implies there exists a unique elliptic solution  $u = u_\sigma \in C^3(\overline{\Omega})$  of the boundary value problem

$$(5.6) \quad \begin{aligned} F[u] &= e^{\sigma u} B(\cdot, Du), \\ Tu(\Omega) &= \Omega^* \end{aligned}$$

for arbitrary  $\sigma > 0$ , with  $Tu$  one-to-one. To complete the proof of Theorem 1.2, (at least when the above foliations exist), we assume that  $B$  satisfies (1.5), (1.8) and (1.9). As above we see that the integrals

$$\int_{\Omega} e^{\sigma u_\sigma}$$

are uniformly bounded, with  $D(\sigma u_\sigma) \rightarrow 0$  as  $\sigma \rightarrow 0$ . Consequently  $\sigma u_\sigma \rightarrow \text{constant} = 0$  by (1.19) and modulo additive constants,  $u_\sigma \rightarrow u$  as  $\sigma \rightarrow 0$ , where  $u$  is the solution of (1.14), (1.7), as required.

We may construct the family of domains  $\{\Omega_t\}$  used above, if we are given a defining function  $C^4$  defining function  $\varphi$ , satisfying

$$(5.7) \quad [D_{ij}\varphi(x) - c^{l,k}_{ij,l}(x, y)D_k\varphi(x)]\xi_i\xi_j \geq \delta_0|\xi|^2$$

for all  $x \in \Omega, y \in \Omega^*, \xi \in \mathbb{R}^n$ , which takes its minimum at  $x_0$ . Note that the uniform  $c$ -convexity of  $\Omega$  implies the existence of a defining function satisfying (5.7) in a neighbourhood of  $\partial\Omega$ , as in (2.26). There are various ways of constructing suitable families from a global defining function,  $\varphi$ . In particular taking  $\varphi(x_0) = -1$ , we may choose

$$\Omega_t = \{x \in \Omega \mid \varphi_t(x) < 0\}$$

where  $\varphi_t$  is defined by

$$(5.8) \quad \varphi_t = t(\varphi - a) + (t_0 - t)\varphi_0,$$

for  $\varphi_0(x) = |x - x_0|^2 - r^2$ ,  $t \leq t_0$ , for some  $0 < t_0 < 1$  and  $a$  close to  $-1$  to first deform to a small sub-level set of  $\varphi$ , followed by taking  $\varphi_t = (1 - t)a/(1 - t_0)$  for  $t \geq t_0$ .

Alternatively, we could have chosen  $\Omega_0 = \varphi < a$  at the outset and only used the second deformation. The domains  $\Omega_t^*$  may be similarly constructed. If the curvatures of  $\partial\Omega^*$  are sufficiently large, for example if  $\Omega^*$  is a small ball, then the existence of a defining function satisfying (5.7) follows by pulling back from a single image  $c_y(\cdot, y_0)(\Omega)$ . As a byproduct, we see that if  $\Omega$  is uniformly  $c$ -convex with respect to a single point  $y_0 \in \Omega^*$ , there at least exists a smooth function  $u_0 \in C^3(\overline{\Omega})$  satisfying the ellipticity condition (1.2). Moreover  $Tu_0$  is a diffeomorphism from  $\Omega$  to a small ball  $B_r(y_0) \subset \Omega^*$ .

From the above considerations, we see that the proof of Theorem 1.2 is completed in the cases where either  $\Omega$  or  $\Omega^*$  is a small ball. The general case will then follow by further use of the method of continuity if there exists a defining function satisfying (5.7) for either domain. However we will take up a different approach from this point and use the function  $u_0$  constructed above to construct a further function  $u_1$  approximately satisfying our given boundary conditions to which the method of continuity can be applied without domain variation. Specifically we will prove

**Lemma 5.1.** *Let the domains  $\Omega$  and  $\Omega^*$  and cost function  $c$  satisfy the hypotheses of Theorem 1.1. Then for any  $\epsilon > 0$ , there exists a uniformly  $c^*$ -convex approximating domain,  $\Omega_\epsilon^*$ , lying within distance  $\epsilon$  of  $\Omega^*$ , and satisfying the corresponding condition (1.19) for fixed  $\delta_0$ , together with a function  $u_1 \in C^4(\overline{\Omega})$  satisfying the ellipticity condition (1.2) and the boundary condition (1.7) for  $\Omega_\epsilon^*$ .*

From Lemma 5.1 we complete the proof of Theorem 1.2. We defer the proof of Lemma 5.1 to Section 7 as the proof will use some of the same ingredients as in our discussion of optimal transportation in the next section. In Section 7, we will also indicate an alternative and direct construction of the function  $u_1$ , which avoids domain variation altogether in the method of continuity.

## 6. Optimal Transportation

The interior regularity of solutions to the optimal transportation problem is considered in [MTW], under conditions A1, A2, A3 and the  $c^*$ -convexity of the target domain  $\Omega^*$ . Our approach is to first show that the Kantorovich potentials are generalized solutions of the boundary value problem (1.14), (1.7) in the sense of Aleksandrov and Bakel'man. The  $c^*$ -convexity of  $\Omega^*$  is used to show the image of the generalized normal mapping lies in  $\overline{\Omega}^*$  and condition A3 is employed to obtain a priori second derivative estimates from which the desired regularity follows. The potential functions  $u$  and  $v$  solve the dual

problem of minimizing the functional

$$(6.1) \quad I(u, v) = \int_{\Omega} f u + \int_{\Omega^*} g v$$

over the set  $K$  given by

$$(6.2) \quad K = \{(u, v) \mid u, v \in C^0(\Omega), C^0(\Omega^*) \text{ resp. } u(x) + v(y) \geq c(x, y) \text{ for all } x \in \Omega, y \in \Omega^*\}.$$

The potential functions  $(u, v)$  satisfy the relations

$$(6.3) \quad \begin{aligned} u(x) &= \sup_{y \in \Omega} \{c(x, y) - v(y)\}, \\ v(y) &= \sup_{x \in \Omega} \{c(x, y) - u(x)\}, \end{aligned}$$

that is they are the  $c^*$  and  $c$  transforms of each other. Since  $c \in C^{1,1}$ , they will be semi-convex. The optimal mapping  $T$  is then given almost everywhere by (1.23) and the equation (1.14) will be satisfied with elliptic solution  $u$  almost everywhere in  $\Omega$ . The functions  $u$  and  $v$  are respectively  $c$  and  $c^*$ -convex. A function  $u \in C^0(\Omega)$  is called  $c$ -convex in  $\Omega$  if for each  $x_0 \in \Omega$ , there exists  $y_0 \in \mathbb{R}^n$  such that

$$(6.4) \quad u(x) \geq u(x_0) + c(x, y_0) - c(x_0, y_0)$$

for all  $x \in \Omega$ . If  $u$  is a  $c$ -convex function, for which the mapping  $T$  given by (1.23) is measure preserving, then it follows that  $u$  is a potential and again  $T$  is the unique optimal mapping. These results hold under the hypotheses A1 and A2 and it suffices to assume the densities  $f, g \geq 0, \in L^1(\Omega), L^1(\Omega^*)$ , respectively, whence the mapping  $T$  is only determined almost everywhere on the set where  $f$  is positive. The reader is referred to [C3,GM,MTW,U2,V] for further details.

From the above discussion we see that the solution of the boundary value problem (1.14), (1.7) will automatically furnish a potential for the optimal transportation problem if it is  $c$ -convex. Note that ellipticity only implies that the solution is locally  $c$ -convex and we need a further argument to conclude the global property, unlike the case of quadratic cost functions and convex solutions. First we recall the concept of generalized solution introduced in [MTW]. Let  $u$  be a  $c$ -convex function on the domain  $\Omega$ . The  $c$ -normal mapping,  $\chi_u$ , is defined by

$$(6.5) \quad \chi_u(x_0) = \{y_0 \in \mathbb{R}^n \mid u(x) \geq u(x_0) + c(x, y) - c(x_0, y_0), \text{ for all } x \in \Omega\}.$$

Clearly,  $\chi_u(x_0) \subset Y(x_0, \partial u(x_0))$  where  $\partial$  denotes the subgradient of  $u$ . For  $g \geq 0, \in L^1_{\text{loc}}(\mathbb{R}^n)$ , the generalized *Monge-Ampère measure*  $\mu[u, g]$  is then defined by

$$(6.6) \quad \mu[u, g](e) = \int_{\chi_u(e)} g$$

for any Borel set  $e \in \Omega$ , so that  $u$  satisfies equation (1.14) in the generalized sense if

$$(6.7) \quad \mu[u, g] = f \, dx .$$

The boundary condition (1.7) is satisfied in the generalized sense if

$$(6.8) \quad \Omega^* \subset \chi_u(\overline{\Omega}), \quad \left| \{ x \in \Omega \mid f(x) > 0 \text{ and } \chi_u(x) - \overline{\Omega}^* \neq \emptyset \} \right| = 0$$

The theory of generalized solutions replicates that for the convex case,  $c(x, y) = x \cdot y$ , [MTW]. If  $f$  and  $g$  are positive, bounded measurable functions on  $\Omega, \Omega^*$  respectively satisfying the mass balance condition (1.9), and  $c$  satisfies A1, A2, then there exists a unique (up to constants) generalized solution of (6.7), (6.8), (with  $g = 0$  outside  $\Omega^*$ ), which together with its  $c$  transform  $v$ , given by (6.3), uniquely solves the dual problem (6.1), (6.2) ([MTW]).

Now let  $u \in C^2(\overline{\Omega})$  be an elliptic solution of the boundary value problem (1.7), (1.14) and  $v$  a  $c$ -convex solution of the corresponding generalized problem. By adding constants, we may assume  $\inf_{\Omega}(u - v) = 0$ . We need to prove  $u = v$  in  $\Omega$ , that is the strong comparison principle holds. Let  $\Omega'$  denote the subset of  $\Omega$  where  $u > v$  and first suppose that  $\partial\Omega' \cap \Omega \neq \emptyset$ . Note that if  $v \in C^2(\Omega)$ , this situation is immediately ruled out by the classical strong maximum principle [GT]. Otherwise we may follow the proof of the strong maximum principle as there will exist a point  $x_0 \in \partial\Omega' \cap \Omega$ , where  $\Omega'$  satisfies an interior sphere condition, that is there exists a ball  $B \subset \Omega - \Omega'$  such that  $x_0 \in \partial\Omega' \cap \partial B$ ,  $u(x_0) = v(x_0)$  and  $u > v$  in  $B$ . Since  $v$  is semi-convex,  $v$  will be twice differentiable at  $x_0$ , with  $Dv(x_0) = Du(x_0)$ . Moreover by passing to a smaller ball if necessary we may assume both  $u$  and  $v$  are  $c$ -convex in  $B$ . Since  $u$  is a smooth elliptic solution of (1.14), there will exist a strict supersolution  $w \in C^2(\overline{B} - B_\rho)$ , for some concentric ball  $B_\rho$  of radius  $\rho < R$ , satisfying  $w(x_0) = u(x_0)$ ,  $w \geq v$  on  $\partial B \cup \partial B_\rho$ ,  $Dw(x_0) \neq Du(x_0)$ . By the comparison principle, [MTW], Lemma 5.2, we have  $w \geq v$  in  $B - B_\rho$ , and hence  $Dw(x_0) = Dv(x_0)$ , which is a contradiction. Thus we may assume  $\partial\Omega' \cap \Omega = \emptyset$ , that is  $u > v$  in  $\Omega$  with  $u(x_0) = v(x_0)$  for some point  $x_0 \in \partial\Omega$ . From our argument above, we obtain a function  $w \in C^2(\overline{B} - B_\rho)$  satisfying  $w(x_0) = u(x_0) = v(x_0)$ ,  $v \leq w \leq u$  in  $\overline{B} - B_\rho$ , together with

$$(6.9) \quad u(x) - w(x) \geq \epsilon |x - x_0|$$

for all  $x \in B_R - B_\rho$ . Since  $v \leq w$  in  $B_R - B_\rho$ , this contradicts the obliqueness condition (2.43) if  $\Omega^*$  is  $c^*$ -convex.

Alternatively we may proceed directly ( and more simply) as follows to show that the solution  $u$  is  $c$ -convex, using the property that  $Tu$  is one-to-one. In fact, as mentioned previously in Section 2, this would follow automatically from the change of variables

formula by virtue of the mass balance condition (1.9). Let  $x_0 \in \Omega$  and  $y_0 = Tu(x_0)$ . Suppose there exists a point  $x_1 \in \Omega$ , where

$$(6.10) \quad u(x_1) < c(x_1, y_0) - c(x_0, y_0).$$

By downwards vertical translation, there exists a point  $x_2 \in \partial\Omega$ , satisfying

$$(6.11) \quad u(x) > u(x_2) + c(x, y_0) - c(x_2, y_0).$$

for all  $x \in \Omega$ . Putting  $y_2 = Tu(x_2)$ , we must also have

$$(6.12) \quad c_x(x_2, y_2) \cdot \gamma(x_2) < c_x(x_2, y_0) \cdot \gamma(x_2),$$

which again contradicts the  $c^*$ -convexity of  $\Omega^*$ .

### Remarks.

In the first proof above, we employed a comparison result that if  $u$  is a classical elliptic supersolution of (1.14) dominating a generalized subsolution  $v$  on the boundary of a subdomain  $\Omega'$ , then  $u \geq v$  in  $\Omega'$ . In our local uniqueness argument in [MTW], we also used implicitly the complementing result that if  $u$  is an elliptic subsolution dominated by a generalized supersolution  $v$  on  $\partial\Omega'$ , then  $u \leq v$  in  $\Omega'$ . However, in this case, we cannot apply Lemma 5.2 in [MTW] directly as local  $c$ -convexity of  $v$  may not imply global  $c$ -convexity in  $\Omega$ , unless  $v \in C^1(\Omega)$ . This situation is rectified in [TW], under the strong A3 hypothesis. However if  $\Omega$  and  $\Omega^*$  lie respectively in domains  $\Omega_0$  and  $\Omega_0^*$  satisfying the hypotheses of Theorem 1.2, with  $\Omega^*$  also  $c^*$ -convex with respect to  $\Omega_0$ , and  $f$  and  $g$  are positive in  $L^1(\Omega)$  and  $L^1(\Omega^*)$  respectively, then it follows directly by approximation from Theorem 1.3 that the local  $c$ -convexity of the potential  $u$  solving the Kantorovich dual problem implies its global  $c$ -convexity. Other results, such as the  $c$ -convexity of the contact set under condition A3w, also follow from Theorem 1.3 by approximation. The reader is referred to Loeper [L] for a full treatment of this approach, including the sharpness of condition A3w for regularity.

## 7. Completion of Proof of Theorem 1.2

In this section, we provide the proof of Lemma 5.1, thereby completing that of Theorem 1.2. For this purpose, we need to draw on a geometric property of  $c$ -convex domains introduced in [TW]. Namely, suppose that  $\Omega$  is uniformly  $c$ -convex, with respect to  $\Omega^*$ , and that the cost function  $c$  satisfies conditions A1, A2 and A3w. Denoting as before the unit outer normal to  $\partial\Omega$  by  $\gamma$ , we see that the level set  $\mathcal{E}$  of the function  $e$ , given by

$$(7.1) \quad e(x) = e_y(x) = c(x, y) - c(x, y_0),$$



passing through  $x_0$ , is tangential to  $\partial\Omega$  at  $x_0$  if

$$(7.2) \quad y = Y(x_0, p_0 + t\gamma_0),$$

for  $t > 0$ ,  $p_0 = c_x(x_0, y_0)$ ,  $\gamma_0 = \gamma(x_0)$  that is,  $y$  lies on the  $c^*$ -segment which is the image under  $Y(x_0, \cdot)$  of the line from  $p_0$  with slope  $\gamma_0$ . Then it follows from [TW] that  $\Omega$  lies strictly on one side of  $\mathcal{E}$ , whence

$$(7.3) \quad c(x, y) - c(x_0, y) < c(x, y_0) - c(x_0, y_0), \quad x \in \Omega.$$

To prove (7.3) directly from (1.19), we take  $x_0 = 0$ , set  $x' = (x_1, \dots, x_{n-1})$  and choose coordinates so that  $\gamma_0 = (0, \dots, -1)$ . By Taylor's formula,

$$e(x) - e(x_0) \leq -tx_n + [A_{ij}(0, p_0 + t\gamma_0) - A_{ij}(0, p_0)]x'_i x'_j + tO(|x||x_n| + |x|^3).$$

Using (1.13), (1.19), condition A3w and again Taylor's formula, we have

$$\begin{aligned} [A_{ij}(0, p_0 + t\gamma_0) - A_{ij}(0, p_0)]x'_i x'_j &\leq -tc^{l,n}c_{ij,l}(x_0, y_0)x'_i x'_j \\ &\leq tD_i\gamma_j(x_0)x'_i x'_j - t\delta_0|x'|^2, \end{aligned}$$

so that,

$$e(x) - e(x_0) \leq -tx_n + tD_i\gamma_j(x_0)x'_i x'_j + tO(|x||x_n| + |x|^3) < 0,$$

for  $x \in \overline{\Omega} - x_0$ , sufficiently small. Consequently  $\Omega$  lies locally, strictly on one side of  $\mathcal{E}$ . We can then verify the global inequality (7.3) by contradiction, as in [TW]. For if (7.3) is violated, the set

$$U_a = \{x \in \partial\Omega; e(x) > a\}$$

contains two disjoint components, for sufficiently small  $a > 0$ . Increasing  $a$ , we see that these components will meet first at a point  $x^* \in \partial\Omega$  at which the level set of the function  $e$  is tangential, contradicting the local inequality at  $x^*$ .

Now, to commence the proof of Lemma 5.1, we take  $u_0$  to be a function as constructed in Section 6, that is  $u_0$  is a smooth uniformly  $c$ -convex function on  $\Omega$ , whose  $c$ -normal mapping  $Tu_0 = Y(\cdot, Du_0)$  has image  $\omega^*$ , which is a  $c^*$ -convex subdomain of  $\Omega^*$ . Here we call a  $c$ -convex function uniformly  $c$ -convex if it also satisfies the ellipticity condition (1.3). We remark that the  $c$ -convexity of  $u_0$  could also have been proved from (7.3), using the  $c$ -convexity of  $\Omega$  and condition A3w, instead of the  $c^*$ -convexity of  $\Omega^*$  which we used in Section 6. Also by approximation, we may assume  $u_0 \in C^\infty(\overline{\Omega})$ . A function  $h$  is called a  $c$ -function if it has the form

$$h(\cdot) = c(\cdot, y_0) + a$$

for  $y \in \mathbb{R}^n$  and some constant  $a$ . When  $c(x, y) = x \cdot y$ , a  $c$ -function is a linear function. Obviously the  $c$ -normal mapping of  $h$  is the constant map,  $Th(x) = y_0$  for all  $x \in \mathbb{R}^n$ . Let

$$(7.4) \quad u_1(x) = \sup\{u_0, h(x)\}, \quad x \in \Omega^\delta,$$

where the sup is taken over the set  $\mathcal{S}$  of  $c$ -functions  $h$  with  $h \leq u_0$  in  $\Omega$ ,  $T_h(\Omega) \subset \Omega^*$  and  $\Omega^\delta = \{x \in \mathbb{R}^n : \text{dist}(x, \Omega) < \delta\}$ , for some  $\delta > 0$ , is a neighbourhood of  $\Omega$ . The following lemma describes the properties of  $u_1$ .

**Lemma 7.1.** *Assume that the cost function  $c$  satisfies A1, A2, A3w and the domains  $\Omega$  and  $\Omega^*$  are uniformly  $c$ -convex with respect to each other. Then, for sufficiently small  $\delta$ , the function  $u_1$  is a  $c$ -convex extension of  $u_0$  to  $\Omega^\delta$ , whose  $c$ -normal image under  $u_1$  is  $\overline{\Omega}^*$ . Moreover for any point  $x \in \Omega^\delta - \overline{\Omega}$ , there exist unique points,  $x_b \in \partial\Omega$ ,  $y_b \in \partial\Omega^*$ , such that  $\chi_{u_1} = y_b$  on the  $c$ -segment,  $\ell_{x_b}$ , joining  $x$  to  $x_b$ , with respect to  $y_b$ , (except at the endpoint  $x_b$ ) with the resultant mappings being  $C^2$  diffeomorphisms from  $\partial\Omega^r$  to  $\partial\Omega$ ,  $\partial\Omega^*$  respectively, for any  $r < \delta$ .*

Lemma 5.1 will follow from Lemma 7.1 by modification of  $u_1$  outside  $\Omega$  and mollification. To prove Lemma 7.1, we first take any  $c$ -function in the set  $\mathcal{S}$ , with  $c$ -normal image  $y \in \Omega^* - \omega^*$  and increase it until its graph meets that of  $u_0$  on  $\overline{\Omega}$  at a point  $x_b$ , which will lie in  $\partial\Omega$ , since  $u_0$  is uniformly  $c$ -convex. Accordingly we obtain a  $c$ -function  $h \in \mathcal{S}$ , given by

$$(7.5) \quad h(x) = h_{x_b, y}(x) = c(x, y) - c(x_b, y) + u_0(x_b).$$

Since  $h \leq u_0$  in  $\Omega$  and  $h(x_b) = u_0(x_b)$ , we see that the point  $y$  must lie on the  $c^*$ -segment  $\ell_{x_b}^*$ , with respect to  $x_b$ , starting at  $y_{0,b} = Tu_0(x_b)$  and given by

$$(7.6) \quad c_x(x_b, \ell_{x_b}^*) = \{Du_0(x_b) + t\gamma(x_b) : t \geq 0\}.$$

Conversely, for any  $x_b \in \partial\Omega$ ,  $y \in \ell_{x_b}^*$ , we have

$$(7.7) \quad h \leq u_0 \quad \text{in } \Omega,$$

by virtue of (7.3), (taking  $x_0 = x_b, y_0 = y_{0,b}$ ). This proves that  $u_1$  is indeed a  $c$ -convex extension of  $u_0$  to  $\Omega^\delta$ .

To proceed further, we let  $y_b$  be the unique point in  $\partial\Omega^*$ , where  $\ell_{x_b}^*$  intersects  $\partial\Omega^*$ . Since  $\omega^*$  is also uniformly  $c^*$ -convex,  $\ell_{x_b}^*$  only intersects  $\partial\omega^*$  at the initial point  $y_{0,b}$ . Actually, only the uniform  $c$ -convexity of  $u_0$  is needed to justify this. Henceforth we restrict  $\ell_{x_b}^*$  to the segment joining  $y_{0,b}$  to  $y_b$ . From our argument above, the mapping

from  $x_b$  to  $y_b$  is onto  $\partial\Omega^*$ . We claim it is also one-to-one. For suppose the  $c$ -function  $h = h_{x_b, y_b}$  meets  $\partial\Omega$  at another point  $x'$ . By increasing  $h$ , we infer the existence of a further boundary point  $x''$ , which is a saddle point for  $u_0$ , contradicting (7.3). It follows then that the mapping from  $x_b$  to  $y_b$  is a  $C^3$  diffeomorphism from  $\partial\Omega$  to  $\partial\Omega^*$ . Next, if  $B_r$  is a sufficiently small tangent ball of  $\Omega$  at  $x_b$ , it will also be uniformly  $c$ -convex so again by (7.3), we obtain

$$(7.8) \quad h_{x_b, y_b}(x) > h_{x_b, y}(x) \quad \forall x \in B_r, y \in \ell_{x_b}^*.$$

and thus we have

$$(7.9) \quad u_1 = \max_{x_b \in \partial\Omega} \{u_0, h_{x_b, y_b}\}, \quad x \in \Omega^\delta.$$

To complete the proof of Lemma 7.1, we need to show that for each  $x \in \Omega^\delta - \Omega$ , there exists a unique  $x_b \in \partial\Omega$  where the maximum in (7.8) is attained. For this purpose, we invoke the  $c$ -transform of  $u_1$ ,

$$(7.10) \quad v_0(y) = \sup\{c(x, y) - u_1(x) : x \in \Omega^\delta\}, \quad y \in \Omega^*,$$

which extends the  $c$ -transform of  $u_0$  in  $\omega^*$ . Moreover, by (7.7), we see that for  $y \in \ell_{x_b}^*$ , the sup is attained at  $x_b$ . Hence

$$(7.11) \quad v_0(y) = c(x_b, y) - u_0(x_b) \quad \forall y \in \ell_{x_b}^*.$$

One easily verifies that  $v_0$  is smooth in  $\overline{\Omega}^* - \partial\omega^*$ . Using (7.11) and arguing as before, we infer that for any point  $x \in \Omega^\delta - \overline{\Omega}$ , there exists a unique point  $y_b \in \partial\Omega^*$  such that

$$(7.12) \quad \begin{aligned} h_{x, y_b}^*(y) &= c(x, y) - c(x, y_b) + v_0(y_b) \\ &\leq v_0(y) \quad \forall y \in \overline{\Omega}^*. \end{aligned}$$

Moreover  $x$  lies on the  $c$ -segment,  $\ell_{y_b}$  given by

$$(7.13) \quad c_y(\ell_{y_b}, y_b) = \{Dv_0(y_b) + t\gamma^*(y_b) : t \in [0, \bar{\delta}]\},$$

where  $\gamma^*$  denotes the unit outer normal to  $\partial\Omega^*$  and  $\bar{\delta}$  is a small constant. Note that  $Dv_0(y_b) = c_y(x_b, y_b)$ . From (7.12), we see that the maximum in (7.9) is attained at  $x_b, y_b$  so

$$(7.14) \quad u_1(x) = c(x, y_b) - c(x_b, y_b) + u_0(x_b) \quad x \in \ell_{y_b},$$

with  $\chi_{u_1}(\ell_{y_b} - \{x_b\}) = y_b$ ,  $\chi_{u_1}(x_b) = \ell_{x_b}^*$ . From the obliqueness of  $\ell_{y_b}$  on  $\partial\Omega$ , we have that the mapping from  $x \in \Omega^r$  to  $x_b$  is one-to-one for sufficiently small  $r$ . This completes the proof of Lemma 7.1.

From Lemma 7.1, we see that the function  $u_1$  is smooth in  $\Omega^\delta - \partial\Omega$ . Furthermore, with  $\delta$  sufficiently small,  $u_1$  will be tangentially uniformly convex on  $\partial\Omega^r$ , that is

$$(7.15) \quad [D_{ij}u_1 - c_{ij}(\cdot, Tu_1)]\tau_i\tau_j \geq \lambda_0,$$

where  $\tau$  is the unit tangent vector on  $\partial\Omega^r$  and  $\lambda_0$  a positive constant. To take care of the normal direction, we modify  $u_1$  in  $\Omega^\delta - \Omega$ , by setting

$$(7.16) \quad u = u_1 + bd^2,$$

where  $b$  is a positive constant and  $d$  denotes distance from  $\Omega$ . Again for  $\delta$  sufficiently small, we infer that  $u$  satisfies

$$(7.16) \quad [D_{ij}u - c_{ij}(\cdot, Tu)]\xi_i\xi_j \geq \lambda_0,$$

in  $\Omega^\delta - \partial\Omega$  for a further positive constant  $\lambda_0$ , and any unit vector  $\xi$ .

We complete the proof of Lemma 5.1 by mollification. Let  $\rho \in C_0^\infty(B_1(0))$  be a mollifier, namely  $\rho$  is a smooth, nonnegative, and radially symmetric function supported in the unit ball  $B_1(0)$  such that the integral  $\int_{B_1(0)} \rho = 1$ . We show that a mollification of  $u$ , given by

$$(7.17) \quad \begin{aligned} u_\varepsilon(x) &= \rho * u \\ &= \int_{\mathbb{R}^n} \varepsilon^{-n} \rho\left(\frac{x-y}{\varepsilon}\right) u(y) dy \\ &= \int_{\mathbb{R}^n} \rho(y) u(x - \varepsilon y) dy \end{aligned}$$

is uniformly c-convex in  $\Omega^{\delta/2}$ , provided  $\varepsilon < \frac{1}{2}\delta$  is sufficiently small and  $x \in \Omega^{\delta/2}$  (so that the value of  $u$  outside  $\Omega^\delta$  is irrelevant). Note that the image of the c-normal mapping of  $u_\varepsilon$  in  $\Omega^{\delta/2}$  is a smooth perturbation of  $\Omega^*$ , and so is also uniformly c-convex provided  $\varepsilon > 0$  is sufficiently small.

It is easy to verify that

$$(7.18) \quad Du_\varepsilon(x) = \int_{\mathbb{R}^n} \rho(y) Du(x - \varepsilon y) dy,$$

$$(7.19) \quad \begin{aligned} D^2u_\varepsilon(x) &= \int_{\mathbb{R}^n} \rho(y) D^2u(x - \varepsilon y) dy + \int_{\partial\Omega} \frac{1}{\varepsilon^n} \rho\left(\frac{x-y}{\varepsilon}\right) \gamma \cdot (D^+u - D^-u)(y) \\ &\geq \int_{\mathbb{R}^n} \rho(y) D^2u(x - \varepsilon y) dy, \end{aligned}$$

where

$$\begin{aligned} D^+u(y) &= \lim_{y' \notin \bar{\Omega}, y' \rightarrow y} Du(y'), \\ D^-u(y) &= \lim_{y' \in \Omega, y' \rightarrow y} Du(y'). \end{aligned}$$

Since,  $\omega^* \in \Omega^*$ , we have

$$(7.20) \quad D_\gamma^+u - D_\gamma^-u \geq C_0 > 0 \quad \text{on } \partial\Omega$$

for some positive constant  $C_0$ . We divide  $\Omega^{\delta/2}$  into three parts:  $\Omega^{\delta/2} = U_1 \cup U_2 \cup U_3$ , where

$$\begin{aligned} U_1 &= \{x \in \Omega^{\delta/2} : \text{dist}(x, \partial\Omega) \geq \varepsilon\}, \\ U_2 &= \{x \in \Omega^{\delta/2} : \text{dist}(x, \partial\Omega) \in ((1 - \sigma)\varepsilon, \varepsilon)\}, \\ U_3 &= \{x \in \Omega^{\delta/2} : \text{dist}(x, \partial\Omega) \leq \varepsilon'\}, \end{aligned}$$

where  $\sigma \in (\frac{1}{2}, 1)$  is a constant close to 1. Since  $u$  is smooth, uniformly c-convex away from  $\partial\Omega$ ,  $u^\varepsilon$  is obviously smooth, uniformly c-convex in  $U_1$  provided  $\varepsilon$  is sufficiently small. By taking  $\sigma > 0$  sufficiently close to 1, for any point  $x_0 \in U_2$ ,  $Du^\varepsilon(x_0)$  is a small perturbation of  $Du(x_0)$ . By (7.19), we also see that the matrix

$$(7.21) \quad \{D^2u^\varepsilon(x_0) - A(x_0, Du^\varepsilon(x_0))\} > 0,$$

namely  $u^\varepsilon$  is smooth, uniformly c-convex in  $U_2$ .

Finally we verify (7.21) in  $U_3$ . For any point  $x_0 \in U_3$ , we choose a coordinate system such that  $x_0 = (0, \dots, 0, x_{0,n})$ , the origin  $0 \in \partial\Omega$  and  $\partial\Omega$  is tangent to  $\{x_n = 0\}$ . To verify (25), we first consider a tangential direction  $\tau$ , namely  $\tau$  is a unit vector tangential to  $\partial\Omega$  at 0. Without loss of generality we assume that  $\tau = (1, 0, \dots, 0)$ . Then we need to prove that

$$(7.22) \quad D_{11}u^\varepsilon(x_0) - A_{11}(x_0, Du^\varepsilon(x_0)) > 0.$$

By our choice of coordinates,  $D_1u^\varepsilon(x_0)$  is a small perturbation of  $D_1u(x_0)$ . Hence it suffices to verify that

$$(7.23) \quad D_{11}u^\varepsilon(x_0) - A_{11}(x_0, D_1u(x_0), D'u^\varepsilon(x_0)) > 0,$$

where  $D'u^\varepsilon = (D_2u^\varepsilon, \dots, D_nu^\varepsilon)$ . By A3w,  $A_{11}$  is convex in  $D'u^\varepsilon$ , whence it follows readily that

$$A_{11}(x_0, D_1u(x_0), D'u^\varepsilon(x_0)) \leq \int_{\mathbb{R}^n} \frac{1}{\varepsilon^n} \rho\left(\frac{x_0 - y}{\varepsilon}\right) A_{11}(x_0, D_1u(x_0), D'u(y)) dy.$$

Inequality (7.22) now follows from (7.19) and (7.23). Note that the argument also applies to any direction  $\eta$  provided  $\eta \cdot \gamma$  is sufficiently small. Next we observe from the second integral in (7.19) that (7.21) holds in the normal direction  $\gamma = e_n$ . Furthermore,

$$(7.24) \quad D_{nn}u^\varepsilon(x_0) - A_{nn}(x_0, Du^\varepsilon(x_0)) \geq K$$

for some  $K$  as large as we want, provided  $\varepsilon$  is sufficiently small. Now suppose the least eigenvalue of the matrix (7.21) is achieved in direction  $\xi$ . We can decompose  $\xi = c_1\tau + c_2e_n$ . If  $c_2 \geq c_0$  for some constant  $c_0 > 0$ , then the matrix (7.21) in direction  $\xi$  is positive by (7.24). Otherwise the proof of (7.22) applies and we also see that the matrix (7.21) in direction  $\xi$  is positive. By appropriate adjustment of  $\Omega$ , we complete the proof of Lemma 5.1 and consequently also Theorems 1.2 and 1.3.

To conclude this section we show that Lemma 5.1 may be proved independently of the arguments in Section 5 by direct construction of a uniformly  $c$ -convex function,  $u_0$ . To do this we let  $y_0$  be a point in  $\Omega^*$  and  $u_0$  be the  $c^*$ -transform of the function

$$(7.25) \quad \psi(y) = -(r^2 - |y - y_0|^2)^{1/2},$$

given by

$$u_0(x) = \sup\{c(x, y) - \psi(y), \quad y \in B_r(y_0)\},$$

for sufficiently small  $r > 0$ . Then  $u_0$  is a locally uniformly  $c$ -convex function defined in some ball  $B_R(0)$ , with  $R \rightarrow \infty$  as  $r \rightarrow 0$ , and the image of its  $c$ -normal mapping,

$$\omega^* := Tu_0(\Omega) \subset B_r(y_0),$$

where  $Tu_0$  is a diffeomorphism between  $\Omega$  and  $\omega^*$ . As  $Tu_0$  is defined on the ball  $B_R(0) \ni \Omega$ ,  $\omega^*$  is a smooth domain. Locally  $u_0$  is a smooth perturbation of the  $c$ -function

$$(7.26) \quad h_0(\cdot) = c(\cdot, y_0) + a_0,$$

for some constant  $a_0$ .

## 8. Examples

We repeat and expand somewhat the examples in [MTW], taking account that our cost functions are the negatives of those there.

### Example 1.

$$(8.1) \quad c(x, y) = -\sqrt{1 + |x - y|^2}$$

Here the vector field  $Y$  and matrix  $A$  are given by

$$(8.2) \quad \begin{aligned} Y(x, p) &= x + \frac{p}{\sqrt{1 - |p|^2}}, \\ A(x, p) &= A(p) = -(1 - |p|^2)^{1/2} (I - p \otimes p). \end{aligned}$$

The cost function satisfies condition A3. We remark that condition A1 is only satisfied for  $|p| < 1$  but this does not prohibit application of our results as the boundedness of target domain  $\Omega^*$  ensures that  $|Du| < 1$  for solutions of (1.7), (1.14). More generally the conditions  $p, q \in \mathbb{R}^n$  in A1 may be replaced by  $p, q \in$  some convex sub-domain.

**Example 2.**

$$(8.3) \quad c(x, y) = -\sqrt{1 - |x - y|^2}$$

Here  $c$  is only defined for  $|x - y| \leq 1$ . The vector field  $Y$  and matrix  $A$  are given by

$$(8.4) \quad \begin{aligned} Y(x, p) &= -x + \frac{p}{\sqrt{1 + |p|^2}}, \\ A(x, p) &= A(p) = (1 + |p|^2)^{1/2} (I + p \otimes p). \end{aligned}$$

The cost function satisfies condition A3. In order to directly apply our results we need to assume  $\Omega$  and  $\Omega^*$  are strictly contained in a ball of radius 1.

**Example 3.** Let  $f, g \in C^2(\Omega), C^2(\Omega^*)$  respectively and

$$(8.5) \quad c(x, y) = x \cdot y + f(x)g(y) .$$

If  $|\nabla f \cdot \nabla g| < 1$ , then  $c$  satisfies A1, A2. If  $f, g$  are convex, then  $c$  satisfies A3w, while if  $f, g$  are uniformly convex, then  $c$  satisfies A3. As indicated in [MTW], the function (8.5) is equivalent to the square of the distance between points on the graphs of  $f$  and  $g$ .

**Example 4. Power costs.**

$$(8.6) \quad c(x, y) = \pm \frac{1}{m} |x - y|^m, \quad m \neq 0, \quad \log|x - y|, \quad m = 0).$$

For  $m \neq 1$  and  $x \neq y$ , when  $m < 1$ , the vector fields  $Y$  and matrices  $A$  are given by

$$(8.7) \quad \begin{aligned} Y(x, p) &= x \pm |p|^{\frac{2-m}{m-1}} p, \\ A(x, p) &= A(p) = \pm \left\{ |p|^{\frac{m-2}{m-1}} I + (m-2) |p|^{-\frac{m}{m-1}} p \otimes p \right\}. \end{aligned}$$

The only cases for which condition A3w is satisfied are  $m = 2(\pm)$  and  $-2 \leq m < 1$  (+ only). For the latter, condition A3 holds for  $-2 < m < 1$ . To apply our results directly in the latter cases, we need to assume  $\Omega$  and  $\Omega^*$  are disjoint.

In [MTW] we also considered the cost function

$$(8.8) \quad c(x, y) = -(1 + |x - y|^2)^{p/2}$$

for  $1 \leq p \leq 2$ , extending Example 1 to  $p > 1$ . We point out here that these functions only satisfy A3 under the restriction  $|x - y|^2 < \frac{1}{p-1}$ . This condition was omitted in [MTW].

### Example 5. Reflector antenna problem

Corresponding results and examples may be obtained on other manifolds such as the spheres  $S^n$ . Indeed the considerations in [MTW] stemmed from the treatment of the reflector antenna problem by Wang in [W1], which may be represented as an optimal transportation problem on the sphere  $S^n$  with cost function

$$(8.9) \quad c(x, y) = \log(1 - x \cdot y),$$

which is simply the spherical analogue of the case  $m = 0$  in Example 4 above. The corresponding vector field  $Y$  is now given by

$$(8.10) \quad Y(x, p) = x - \frac{2}{1 + |p|^2}(x + p),$$

where now  $p$  belongs to the tangent space of  $S^n$  at  $x$ , while the matrix  $A$  is given by

$$(8.11) \quad A = \frac{1}{2}(|p|^2 - 1)g_0 - p \times p,$$

where  $g_0$  denotes the metric on  $S^n$ . See [W1, W2, GW, MTW] for more details. When the domains  $\Omega$  and  $\Omega^*$  have disjoint closures, and spherically uniformly convex boundaries, we obtain the global regularity of potentials.

We will defer further examination and extensions to intersecting domains and other cost functions in a future work. We also point out here that Example 4 provides regularity for quadratic cost functions on spheres when the points  $x$  and  $y$  are sufficiently close.



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